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Optimal Control of Damped Flexible Gyroscopic Systems

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An optimal control scheme for damped gyroscopic systems is presented. The procedure works with real quantities alone. The optimal control scheme is based on the concept of independent modal-space control, leading to a set of independent second-order matrix Riccati equations. The solution of such equations can be obtained with relative ease for both the steady-state and the transient case. Damping is shown to be beneficial, resulting in enhanced control with less control effort.

I. Introduction

THE motion of flexible spececraft can be described by a hybrid set of simultaneous differential equations. Indeed, rigid-body motions are described by ordinary differential equations and elastic motions by partial differential equations. Control of hybrid systems is not within the state of the art. It is customary to transform the hybrid set of differential equations into a set of simultaneous ordinary differential equations by means of a discretization process, such as lumping or a Rayleigh-Ritz type approach, where the latter includes the finite element method.[1] Implicit in the discretization process is also truncation, so that the set of differential equations is finite. The discretized system is only an approximation of the actual system. For the discretized system, the order of the system can at times be relatively large (>100).

The motion of a spacecraft in the neighborhood of equilibrium can be described by a set of linear ordinary differential equations with constant coefficients. For a damped gyroscopic system, the constant coefficients can be arranged into a symmetric mass matrix, a symmetric damping matrix, a skew symmetric gyroscopic matrix, and a symmetric stiffness matrix, where the latter can be assumed to be positive definite. In the undamped case, the open-loop eigenvalue problem can be cast into a real symmetric positive definite form, for which many efficient computational algorithms exist.[1] In the presence of damping, the eigenvalue problem can no longer be cast into a real symmetric positive definite form, so that one is faced with the task of solving the eigenvalue problem for a general (albeit real) matrix. The eigenvalues are generally complex, and difficult to obtain for high-order systems.

The problem of control of undamped flexible gyroscopic systems has been discussed in Refs. 2-5. Using the concept of control in modal space, Ref. 5 describes a Luenberger-type observer that is not plagued by observation spillover instability, as opposed to the case of coupled controls.[6] The results of Ref. 5 have been confirmed in Ref. 7. This paper extends the ideas of Ref. 5 to the damped case.

Damping has not received much attention in the control of flexible spacecraft. Damping in a structure is treated specifically in Refs. 8 and 9, where the idea of providing additional damping through "member damping" is discussed. In both Refs. 8 and 9, the modal damping is provided through velocity feedback. The idea of damper-augmented structures is discussed also in Ref. 10, in which actuators with limited authority are used to produce small damping in an otherwise undamped system. Regarding the actuator-produced damping as a perturbation, first-order changes in the system eigenvalues are computed.

In designing proportional feedback control, one must have a rational way of selecting the control gains. One such method is optimal control using a quadratic performance index, which requires the solution of a matrix Riccati equation.[11] Computational difficulties, however, limit such solutions to matrix equations of low order (<40). But, if independent modal-space control is used,[12] then no such difficulties arise, and instead of solving a coupled matrix Riccati equation of high order one need solve only a set of independent second-order Riccati equations. In fact, if the system is undamped, one can obtain a closed-form solution of the steady-state Riccati equation.

This paper combines and extends the concepts of Refs. 5 and 12 to the optimal control of damped flexible gyroscopic systems. The object is to provide both internal and external decoupling, thus permitting independent control of the modes. Independent control can be achieved only if control is designed in the modal space, which is an abstract space obtained from the actual space via a linear transformation. The optimal control problem is reduced to the solution of n independent 2×2 matrix Riccati equations. The control implementation is carried out in the actual space and it requires as many actuators as the number of degrees of freedom of the system. The method developed in this paper is particularly suited for high-order systems. The only limitation on the system order is that imposed by the capability of the eigensolution algorithms for damped gyroscopic systems. When damping is relatively small, as is the case with most spacecraft, a perturbation approach based on the eigensolution for undamped gyroscopic (and nongyroscopic) systems can be quite effective in producing an eigensolution for damped gyroscopic systems of relatively high order.[13]

II. Decoupling of the Equations of Motion

The equations of motion of a linear damped gyroscopic system can be written in the matrix form[1]

$$m\ddot{q}(t)+(c+g)\dot{q}(t)+kq(t)=Q(t) \tag{1}$$

where m, c, and k are $n\times n$ real symmetric matrices, g is an $n\times n$ real skew symmetric matrix, $q(t)$ is the associated n-dimensional configuration vector, and $Q(t)$ is the associated n-dimensional force vector, which includes the control forces. The mass matrix m is positive definite, the stiffness matrix k is assumed to be positive definite, and the damping matrix c is positive semidefinite.

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Equation (1) represents a set of n second-order ordinary differential equations with constant coefficients in terms of the configuration vector $q(t)$. For the control task, it is more convenient to work with the state vector, which implies transforming the set of n second-order equations into a set of $2n$ first-order equations. Hence, let us introduce the $2n$-dimensional state vector $x(t)$ and associated force vector $X(t)$

$$x(t)=[\dot{q}^T(t) \mid q^T(t)]^T \qquad X(t)=[Q^T(t) \mid 0^T]^T \qquad (2a,b)$$

so that Eq. (1) can be transformed into

$$M\dot{x}(t)+Kx(t)=X(t) \qquad (3)$$

where

$$M=\begin{bmatrix} m & 0 \\ 0 & k \end{bmatrix} \qquad K=\begin{bmatrix} c+g & k \\ -k & 0 \end{bmatrix} \qquad (4a,b)$$

The $2n \times 2n$ matrix M is real symmetric and positive definite and the $2n \times 2n$ matrix K is real but otherwise arbitrary.

Equation (3) can be reduced to standard form by premultiplying both sides of the equation by M^{-1}. Matrix inversions, however, are undesirable from a computational viewpoint. Moreover, in future developments we shall strive to retain symmetry and skew symmetry of coefficient matrices, so that we shall consider the Cholesky decomposition

$$M=LL^T \qquad (5)$$

where L is a lower triangular matrix. Because M is positive definite, such a decomposition is always possible.[1] Introducing Eq. (5) into Eq. (3), considering the linear transformation

$$u(t)=L^Tx(t) \qquad x(t)=L^{-T}u(t) \qquad (6)$$

where $L^{-T}=(L^T)^{-1}=(L^{-1})^T$, and premultiplying the result by L^{-1}, we obtain the standard form

$$\dot{u}(t)=Au(t)+U(t) \qquad (7)$$

where

$$A=-L^{-1}KL^{-T} \qquad U(t)=L^{-1}X(t) \qquad (8a,b)$$

This procedure involves the inverse of L. In this regard, it should be pointed out that the inversion of a triangular matrix is a much simpler operation than the inversion of a fully populated matrix.

We shall seek a solution of Eq. (7) in modal space, which implies deriving the Jordan form associated with the coefficient matrix A. To this end, we assume that the modal vectors of A are independent, in which case the Jordan matrix is diagonal. Actually, we shall opt for a formulation in terms of real quantities, of that the Jordan matrix will have a special form, namely a block-diagonal form, with the blocks consisting of 2×2 matrices. This results in a set of n second-order systems for pairs of conjugate modal coordinates. If the feedback control forces are designed properly, then the n second-order systems are independent. The control forces designed in modal space are then synthesized to produce the actual controls. If the pseudoJordan form just described can be obtained without much difficulty, then control in modal space has clear computational advantages over coupled controls.

Let us consider the eigenvalue problem

$$Au_i=\lambda_iu_i \qquad (i=1,2,\ldots,2n) \qquad (9a)$$

Table 1 Open-loop poles

r \ $\epsilon=$	0	0.01	0.05
1	$0 \pm i1.6694$	$-0.0092 \pm i1.6695$	$-0.0460 \pm i1.6702$
2	$0 \pm i3.3108$	$-0.0331 \pm i3.3106$	$-0.1655 \pm i3.3066$
3	$0 \pm i4.0279$	$-0.0144 \pm i4.0280$	$-0.0720 \pm i4.0291$
4	$0 \pm i5.9125$	$-0.0625 \pm i5.9121$	$-0.3125 \pm i5.9021$
5	$0 \pm i8.5251$	$-0.0584 \pm i8.5247$	$-0.2920 \pm i8.5124$
6	$0 \pm i14.389$	$-0.1439 \pm i14.388$	$-0.7194 \pm i14.371$
7	$0 \pm i15.209$	$-0.1654 \pm i15.208$	$-0.8271 \pm i15.186$
8	$0 \pm i19.584$	$-0.1994 \pm i19.583$	$-0.9972 \pm i19.559$

where A is a real arbitrary matrix, as well as the adjoint eigenvalue problem

$$A^Tv_i=\lambda_iv_i \qquad (i=1,2,\ldots,2n) \qquad (9b)$$

in which λ_i are the eigenvalues of A, and hence the eigenvalues of A^T as well, and u_i and v_i are the right and left eigenvectors of A, respectively. The eigenvectors are biorthogonal and can be normalized so as to satisfy

$$v_j^Tu_i=2\delta_{ij} \qquad v_j^TAu_i=2\lambda_i\delta_{ij} \qquad (i,j=1,2,\ldots,2n) \qquad (10)$$

where δ_{ij} is the Kronecker delta. The reason for the factor 2 on the right side of Eqs. (10) will become evident shortly.

The eigenvalues λ_j and the eigenvectors u_j and v_j are generally complex quantities and they occur in pairs of complex conjugates, or

$$\lambda_j=\alpha_j+i\beta_j \qquad \lambda_{j+n}=\bar{\lambda}_j=\alpha_j-i\beta_j$$
$$u_j=y_j+iz_j \qquad u_{j+n}=\bar{u}_j=y_j-iz_j$$
$$v_j=r_j+is_j \qquad v_{j+n}=\bar{v}_j=r_j-is_j \quad (j=1,2,\ldots,n) \qquad (11)$$

Our object is to work with real quantities. Hence, introducing the $2n \times 2n$ real modal matrices

$$U=[y_1 \; z_1 \; y_2 \; z_2 \ldots y_n \; z_n]$$
$$V=[r_1 \; -s_1 \; r_2 \; -s_2 \ldots r_n \; -s_n] \qquad (12)$$

the biorthonormality relations, Eqs. (10), can be written in the compact matrix form

$$V^TU=I \qquad V^TAU=A^* \qquad (13)$$

where I is the identity matrix of order $2n$ and

$$A^*=\text{block-diag } A_r^* \qquad (14a)$$

is the block-diagonal pseudoJordan form mentioned earlier, in which

$$A_r^*=\begin{bmatrix} \alpha_r & \beta_r \\ -\beta_r & \alpha_r \end{bmatrix} \qquad (r=1,2,\ldots,n) \qquad (14b)$$

are the 2×2 blocks. Equations (13) explain the factor 2 on the right side of Eqs. (10).

Next, let us consider the $2n \times 2n$ real matrices P and Q defined by

$$P=L^{-T}U \qquad Q=L^{-T}V \qquad (15)$$

Introducing Eqs. (15) into Eqs. (13) and considering Eqs. (5) and (8a), we conclude that

$$Q^TMP=I \qquad Q^TKP=-A^* \qquad (16a,b)$$

In view of the foregoing, let us consider the real modal state vector $w(t)$ given by the linear transformation

$$x(t)=Pw(t) \tag{17}$$

Inserting Eq. (17) into Eq. (3), premultiplying the result by Q^T and recalling Eqs. (16), we obtain

$$\dot{w}(t)=A^*w(t)+W(t) \tag{18}$$

where

$$W(t)=Q^TX(t) \tag{19}$$

is a real generalized control vector. Since P and Q have full rank, the system of Eqs. (18) and (19) is controllable. Note that the control vector $X(t)$ can be obtained from the generalized control vector $W(t)$ by means of

$$X(t)=MPW(t) \tag{20}$$

Equation (18) represents a set of n pairs of first-order differential equations of the type

$$\dot{w}_r(t)=A_r^*w_r(t)+W_r(t) \qquad (r=1,2,...,n) \tag{21}$$

where

$$w_r(t)=[\xi_r(t)\ \ \eta_r(t)]^T \quad W_r(t)=[W_{\xi r}(t)\ \ W_{\eta r}(t)]^T \tag{22}$$

are two-dimensional vectors and A_r^* are 2×2 matrices as given by Eqs. (14b).

Equations (21) have the appearance of independent equations, but this may be misleading. Indeed, when the generalized feedback forces $W_r(t)$ depend on generalized coordinates other than $w_r(t)$, then the decoupling is only *internal*. On the other hand, if $W_r(t)$ depends only on $w_r(t)$, i.e., if

$$W_r=W_r(w_r) \qquad (r=1,2,...,n) \tag{23}$$

then *external decoupling* is also achieved, so that decoupling is *complete*, thus permitting *independent control* of each mode. The modal-space control of Ref. 5 is essentially a modal control method in which complete decoupling of the controlled modes is achieved.

The modal-space control scheme just described is based on the solution of the open-loop eigenvalue problem associated with the matrix A, Eq. (9a), as well as of the adjoint eigenvalue problem, Eq. (9b). Note that these eigensolutions, used in conjunction with Eqs. (15-17), are those that permitted the internal decoupling mentioned earlier. For arbitrary matrices A, these eigensolutions are generally complex, even when A is real. For high-order systems (>100), such eigensolutions are likely to be computationally unattractive. But, for the most part, damping is relatively small, so that some advantage can be derived from this fact. Indeed, the eigensolution can be produced with greatly increased efficiency by resorting to a perturbation technique that bases the eigensolution for damped gyroscopic systems on the eigensolution for real symmetric matrices, as shown in the Appendix.

III. Control in Modal Space

In Ref. 5, a procedure was developed whereby n (or less) of the state modes are controlled and the remaining ones are left uncontrolled. The same approach will be adopted here. Hence, let us write the modal state vector and the associated modal control vector in the form

$$w=[w_C^T \mid w_R^T]^T \qquad W=[W_C^T \mid W_R^T]^T \tag{24a,b}$$

where the subscripts C and R refer to controlled and residual quantities.

Similarly, we partition the coefficient matrix A^* as follows:

$$A^*=\left[\begin{array}{c|c} A_C^* & 0 \\ \hline 0 & A_R^* \end{array}\right] \tag{25}$$

where A_C^* and A_R^* are $n\times n$ matrices. In view of Eq. (14b), this requires that n be even. Introducing Eqs. (24) and (25) into Eq. (18), we obtain

$$\dot{w}_C=A_C^*w_C+W_C \qquad \dot{w}_R=A_R^*w_R+W_R \tag{26}$$

Next, let us write the matrix P in the partitioned form

$$P=\left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] \tag{27}$$

where P_{ij} $(i,j=1,2)$ are $n\times n$ matrices. Then, considering Eqs. (2b), (4a), (20), (24b) and (27), we obtain

$$Q=m(P_{11}W_C+P_{12}W_R) \qquad 0=k(P_{21}W_C+P_{22}W_R) \tag{28a,b}$$

Equation (28b) requires that

$$W_R=-P_{22}^{-1}P_{21}W_C \tag{29}$$

so that, introducing Eq. (29) into Eq. (28a), we obtain the relation

$$Q=m(P_{11}-P_{12}P_{22}^{-1}P_{21})W_C \tag{30}$$

Note that, because the columns of P_{22} are independent, P_{22} is guaranteed to be nonsingular. The validity of this statement can be verified by recalling that the system eigenvectors are state vectors, so that the upper halves are equal to the lower halves multiplied by the corresponding eigenvalues. Equation (30) implies that implementation of independent modal space control requires as many actuators as the number of degrees of freedom of the system.

We shall consider proportional control, so that the relation between the modal control vector and modal state vector has the form

$$W=Fw \tag{31}$$

where F is the modal gain matrix. We shall partition F as follows:

$$F=\left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right] \tag{32}$$

where F_{ij} $(i,j=1,2)$ are $n\times n$ matrices. Then, considering Eqs. (24), (29), (31) and (32), it can be shown that the submatrices F_{ij} of F must satisfy

$$F_{21}=-P_{22}^{-1}P_{21}F_{11} \qquad F_{22}=-P_{22}^{-1}P_{21}F_{12} \tag{33}$$

If the residual modes are not to affect the controlled modes, then we must have

$$F_{12}=F_{22}=0 \tag{34}$$

Introducing Eqs. (31), (32), and (34) into Eqs. (26), the closed-loop modal state equations can be written in the matrix form

$$\left[\begin{array}{c} \dot{w}_C \\ \hline \dot{w}_R \end{array}\right]=\left[\begin{array}{c|c} A_C^*+F_{11} & 0 \\ \hline F_{21} & A_R^* \end{array}\right]\left[\begin{array}{c} w_C \\ \hline w_R \end{array}\right] \tag{35}$$

If F_{11} is a fully populated matrix, then all the modal coordinates are recoupled by the feedback control forces. To achieve external decoupling of the controlled modes, and hence complete decoupling of these modes, the feedback control forces must satisfy Eq. (23). This can be accomplished by choosing the matrix F_{11} in the block-diagonal form

$$F_{11} = \text{block-diag}\,[F_r] \tag{36}$$

where

$$F_r = \begin{bmatrix} F_{r11} & F_{r12} \\ F_{r21} & F_{r22} \end{bmatrix} \qquad (r=1,2,\ldots,n/2) \tag{37}$$

are 2×2 matrices. Note that the matrix F_{21} in Eq. (35) will not be block-diagonal. However, because the vector $F_{21}w_C$ acts as a known excitation on the equation for w_R, this latter equation can also be regarded as being in decoupled form.

For future reference, we shall write the relation between the modal controls and the modal coordinates for the controlled modes in the form

$$W_r = F_r w_r \qquad (r=1,2,\ldots,n/2) \tag{38}$$

where W_r and w_r are the two-dimensional vectors given by Eqs. (22).

IV. Independent Optimal Control in Modal Space

The question as to how to determine the control gain matrix F_{11} remains. We shall determine F_{11} so that the control be optimal in the sense that a certain quadratic cost function is minimized. We shall write the cost function in the form

$$J_C = \sum_{r=1}^{n/2} J_r \tag{39}$$

where J_r are modal cost functions. The number of functions is, of course, equal to the number of controlled modes. Because each mode is controlled independently, the modal cost functions J_r are independent of one another. It follows that J_C can be minimized by minimizing each and every J_r separately.

We shall take a typical cost function in the form[12]

$$J_r = \tfrac{1}{2}[w_r(t_f) - \hat{w}_r]^T H_r [w_r(t_f) - \hat{w}_r] + \tfrac{1}{2}\int_{t_0}^{t_f} [w_r^T(t) Q_r w_r(t) + W_r^T(t) R_r W_r(t)]\,\mathrm{d}t \qquad (r=1,2,\ldots,n/2) \tag{40}$$

where $\hat{w}_r$ is a reference value to which $w_r(t)$ is to be driven at the final time t_f. Moreover, H_r, Q_r, and R_r are real symmetric 2×2 matrices, the first two being positive semidefinite and the third being positive definite. Of course, the vector $W_r(t)$ is related to the vector $w_r(t)$ by the matrix F_r, as indicated by Eq. (38), so that independent control for each mode can indeed be realized.

We shall consider the regulator problem, $\hat{w}_r = \mathbf{0}$. In addition, we shall choose $H_r = 0$ and $Q_r = I$, so that only the matrix R_r remains to be selected. Note that if $Q_r = I$, then the term $w_r^T Q_r w_r$ represents the total energy in the rth mode. Any other weighting can be easily absorbed in R_r. In can be shown[12] that the optimal modal control gain matrix can be written in the form

$$F_r(t) = -R_r^{-1} K_r(t) \qquad (r=1,2,\ldots,n/2) \tag{41}$$

where $K_r(t)$ is a 2×2 real symmetric matrix satisfying the matrix Riccati equation

$$\dot{K}_r(t) = -K_r(t)A_r^* - A_r^{*T}K_r(t) - I + K_r(t)R_r^{-1}K_r(t) \qquad (r=1,2,\ldots,n/2) \tag{42}$$

It turns out that the two-dimensional modal vector $w_r(t) = [\xi_r(t) \;\; \eta_r(t)]^T$ can be controlled by only one component of the modal force vector $W_r(t) = [W_{\xi r}(t) \;\; W_{\eta r}(t)]^T$. We shall choose $W_{\xi r} = 0$. The same effect can be achieved by choosing

$$R_r = \begin{bmatrix} \infty & 0 \\ 0 & R_{\eta r} \end{bmatrix} \qquad (r=1,2,\ldots,n/2) \tag{43}$$

where we note that we do not actually work with R_r but with its inverse

$$R_r^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & R_{\eta r}^{-1} \end{bmatrix} \qquad (r=1,2,\ldots,n/2) \tag{44}$$

Indeed, letting

$$K_r = \begin{bmatrix} K_{r11} & K_{r12} \\ K_{r21} & K_{r22} \end{bmatrix} \qquad (r=1,2,\ldots,n/2) \tag{45}$$

and inserting Eqs. (44) and (45) into Eq. (42), we conclude that the matrix F_r reduces to

$$F_r = -R_{\eta r}^{-1} \begin{bmatrix} 0 & 0 \\ K_{r21} & K_{r22} \end{bmatrix} \qquad (r=1,2,\ldots,n/2) \tag{46}$$

Introducing Eq. (46) into Eq. (38), we can write explicitly

$$W_{\xi r} = 0 \qquad W_{\eta r} = -R_{\eta r}^{-1}(K_{r21}\xi_r + K_{r22}\eta_r) \qquad (r=1,2,\ldots,n/2) \tag{47}$$

The value of $R_{\eta r}$ must be selected by the analyst. When this is done, the values of K_{r21} and K_{r22} are obtained by solving Riccati's equation, Eq. (42). Recalling the special form of A_r^*, as well as the fact that $K_r(t)$ is symmetric, Eq. (42) yields

$$\begin{aligned} \dot{K}_{r11} &= -2\alpha_r K_{r11} + 2\beta_r K_{r12} - 1 + R_{\eta r}^{-1}K_{r12}^2 \\ \dot{K}_{r12} &= -\beta_r K_{r11} - 2\alpha_r K_{r12} + \beta_r K_{r22} + R_{\eta r}^{-1}K_{r12}K_{r22} \\ \dot{K}_{r22} &= -2\beta_r K_{r11} - 2\alpha_r K_{r12} - 1 + R_{\eta r}^{-1}K_{r22}^2 \end{aligned} \tag{48}$$

There are several solutions of Eqs. (48) of special interest.

A. Steady-State Solution

If the system is completely controllable and if $H_r = 0$ and A_r^*, Q_r, and R_r are constant matrices, then $\lim_{t_f\to\infty} K_r(t) = K_r = \text{const.}$[11] Of course, in this case $\dot{K}_{r11} = \dot{K}_{r12} = \dot{K}_{r22} = 0$, so that Eqs. (48) reduce to the nonlinear algebraic equations

$$\begin{aligned} -2\alpha_r K_{r11} + 2\beta_r K_{r12} - 1 + R_{\eta r}^{-1}K_{r12}^2 &= 0 \\ -\beta_r K_{r11} - 2\alpha_r K_{r12} + \beta_r K_{r22} + R_{\eta r}^{-1}K_{r12}K_{r22} &= 0 \\ -2\beta_r K_{r11} - 2\alpha_r K_{r12} - 1 + R_{\eta r}^{-1}K_{r22}^2 &= 0 \end{aligned} \tag{49}$$

The solution of Eqs. (49) is known as the steady-state solution.

In the undamped case, $\alpha_r = 0$, the solution of Eqs. (49) can be obtained with relative ease.[12] In the damped case, $\alpha_r \neq 0$, the solution is more difficult to obtain, but it is still manageable. Solving the first and third of Eqs. (49) for K_{r11} and K_{r12} and substituting the result into the second, we obtain a polynomial of degree four in K_{r22}. Exact roots of a fourth-degree polynomial can be computed[14] but the procedure is tedious. The solution was obtained here by an iterative technique based on Newton's method.

B. Transient Solution

Riccati's equation, Eq. (42), is a 2×2 matrix nonlinear ordinary differential equation. It can be transformed into a 4×4 matrix linear equation.[12] To this end, we write the transient solution of the Riccati equation in the form

$$K_r(t_r)=E_r(t_r)L_r^{-1}(t_r) \tag{50}$$

where $t_r=t_f-t$ is the remaining time. Equation (50) is subject to the initial conditions

$$E_r(t_r=0)=H_r \qquad L_r(t_r=0)=I \tag{51}$$

Introducing Eq. (50) into Eq. (42) and considering Eqs. (51), the nonlinear Riccati equation can be replaced by a linear equation of twice the order of the form

$$\begin{bmatrix} E_r'(t_r) \\ L_r'(t_r) \end{bmatrix} = M_r \begin{bmatrix} E_r(t_r) \\ L_r(t_r) \end{bmatrix} \tag{52}$$

where primes denote differentiation with respect to t_r and in which

$$M_r=\left[\begin{array}{c|c} A_r^{*T} & Q_r \\ \hline R_r^{-1} & -A_r^* \end{array}\right] \tag{53}$$

For the damped gyroscopic system under consideration

$$M_r=\begin{bmatrix} \alpha_r & -\beta_r & 1 & 0 \\ \beta_r & \alpha_r & 0 & 1 \\ 0 & 0 & -\alpha_r & -\beta_r \\ 0 & R_{\eta r}^{-1} & \beta_r & -\alpha_r \end{bmatrix}$$

To implement the modal-space controllers, it is necessary to have estimates of the modal state $w(t)$. To this end, one can construct a modal observer.[5] It is shown in Ref. 5 that if the observer is of order n and the output vector is n-dimensional, then no observation spillover instability exists. In this paper, we assume that the system is fully observable.

V. Numerical Example

As a numerical example, an eight-degree-of-freedom dual-spin flexible spacecraft with a despun section has been considered. The spacecraft is shown in Fig. 1. The configuration vector consists of $q=[\theta_1\ \theta_2\ \zeta_1\zeta_2 \ldots,\ \zeta_6]^T$, where θ_1 and θ_2 are nutation angles and $\zeta_1,\zeta_2,\ldots,\zeta_6$ are elastic displacement amplitudes. The first three correspond to the lowest out-of-plane bending, in-plane bending, and torsional deformation modes of the appendage and the next three to the next lowest out-of-plane bending, in-plane bending, and torsional deformation modes. The mathematical model and system parameters are the same as in Refs. 3 and 5. In addition the damping matrix has been taken as diagonal with the entries

$$c_{ii}=2\epsilon(m_{ii}k_{ii})^{1/2} \qquad (i=3,4,\ldots,8)$$

where ϵ is a damping parameter. Using the data from Ref. 3, we can write in terms of ϵ

$$c=\epsilon[0\ 0\ 1.9961\ 1.2082\ 0.7211\ 0.2802\ 0.2800\ 0.2087]$$

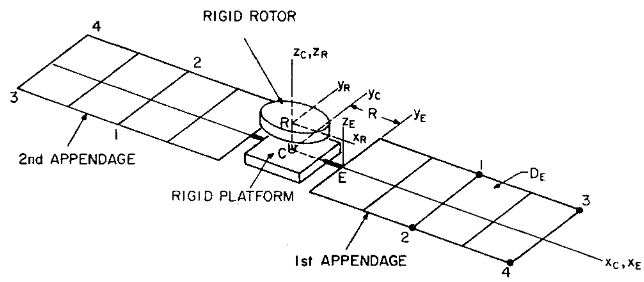

Fig. 1 The mathematical model.

The eigensolution for the damped system was obtained by the perturbation approach described in the Appendix. Whereas for the relatively low-order system of this example the same results could have been obtained by using an eigensolution algorithm for an arbitrary real matrix, for high-order systems this latter algorithm may not be sufficiently reliable.

Using the results of the perturbation eigensolution, the closed-loop poles and the control gains were computed for each of the controlled modes for various values of the damping parameters ϵ. Of the system modes, the second and sixth modes correspond to the in-plane mode, which are not gyroscopic and are uncoupled from the remaining ones. They were ignored in the control problem. Of the remaining modes, the first, third, and fourth modes were controlled and the fifth, seventh, and eighth modes were left uncontrolled.

To implement the control scheme six velocity (or displacement) measurements were needed. They were taken as the two nutation rates $\dot{\theta}_1$, $\dot{\theta}_2$ and velocities in the z direction from four sensors, located on one of the elastic appendages (see points 1, 2, 3, and 4 of Fig. 1). The actuators consisted of two torquers on the central rigid platform and four thrusters at the same locations as the sensors.

Table 1 shows the open-loop eigenvalues of the dynamical system for $\epsilon=0$, 0.01 and 0.05. Table 2 displays the modal control matrices F_r, Eq. (46), that minimize the corresponding modal cost functions for $R_{\eta r}=20$, $r=1,3,4$. Table 3 compares the real parts of the closed-loop poles for the aforementioned values of ϵ and $R_{\eta r}$. As can be observed from Tables 2 and 3, the control gains decrease in magnitude and the real parts of the closed-loop poles increase in magnitude as damping is increased. This implies that, as damping increases, less control effort is necessary and, at the same time, the

Table 2 Control gains matrices

r \ $\epsilon=$	0	0.01	0.05
1	$\begin{bmatrix} 0 & 0 \\ -0.0149 & -0.3159 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -0.0133 & -0.2980 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -0.0084 & -0.2370 \end{bmatrix}$
3	$\begin{bmatrix} 0 & 0 \\ -0.0062 & -0.3162 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -0.0052 & -0.2886 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -0.0026 & -0.2034 \end{bmatrix}$
4	$\begin{bmatrix} 0 & 0 \\ -0.0042 & -0.3162 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -0.0020 & -0.2150 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -0.0002 & -0.0754 \end{bmatrix}$

Table 3 Real parts of closed-loop poles

r \ $\epsilon=$	0	0.01	0.05
1	−0.1580	−0.1582	−0.1645
3	−0.1581	−0.1587	−0.1737
4	−0.1581	−0.1701	−0.3502

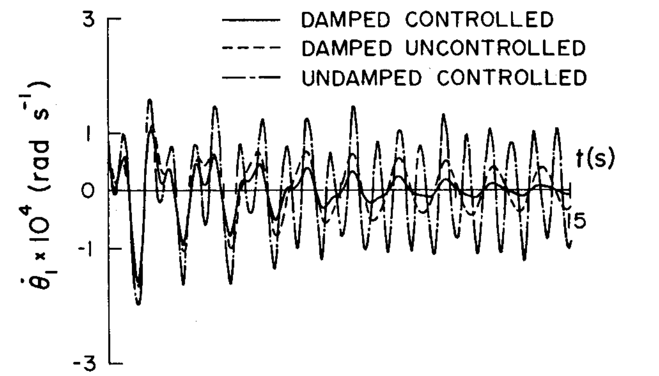

Fig. 2 Time history of the nutational velocity $\dot{\theta}_1$.

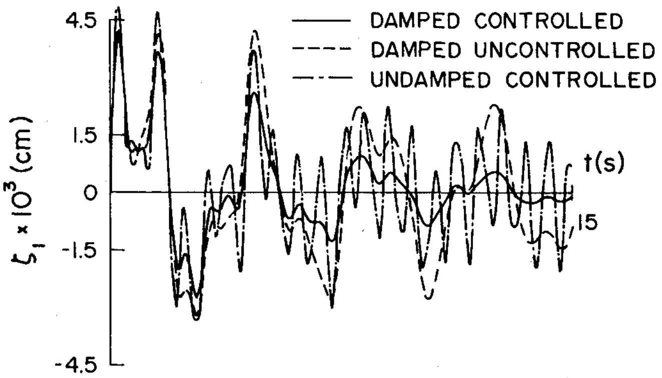

Fig. 3 Time history of the first out-of-plane bending mode.

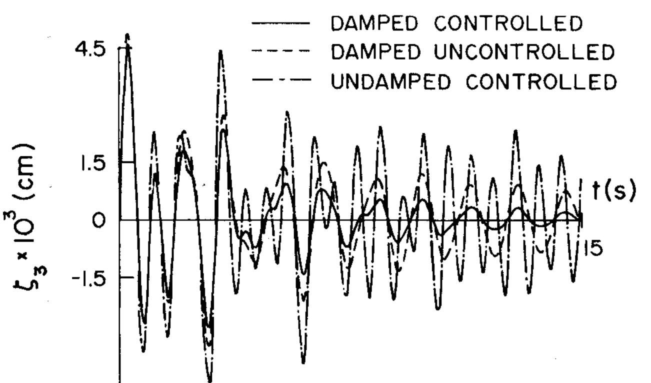

Fig. 4 Time history of the first torsional mode.

amplitudes of the controlled modes reduce to zero at an increasingly faster rate.

The system response was obtained for the following initial conditions:

$$\dot{\theta}_1 = \dot{\theta}_2 = 10^{-4} \qquad \dot{\zeta}_1 = \dot{\zeta}_3 = 2\times 10^{-2} \qquad \dot{\zeta}_4 = \dot{\zeta}_6 = 2\times 10^{-3}$$

$$\dot{\zeta}_2 = \dot{\zeta}_5 = \theta_1 = \theta_2 = \zeta_1 = \zeta_2 = \ldots = \zeta_6 = 0$$

Figures 2-4 show the time histories of $\dot{\theta}_1$, ζ_1, ζ_3 for the damped uncontrolled, undamped controlled, and damped controlled cases. The plots were for $\epsilon = 0.05$. As expected, the addition of damping reduces the control effort considerably and speeds up convergence at the same time.

VI. Summary and Conclusions

An optimal control scheme for damped gyroscopic systems is presented. The procedure works with real quantities alone. The optimal control scheme is based on the concept of independent modal-space control, leading to a set of independent second-order matrix Riccati equations. The solutions of such equations can be obtained with relative ease for both the steady-state and the transient case. This approach eliminates the problem of obtaining an eigensolution for an arbitrary real matrix, where the eigensolution is known in general to be complex. More importantly, it eliminates the problem of solving a high-order Riccati equation. When damping is small it can be treated as a perturbation on the undamped system, permitting an eigensolution for the damped system based on the eigensolution for the undamped system, where the latter can be obtained readily by efficient algorithms. Damping is shown to be beneficial, resulting in enhanced control with less control effort.

Appendix: Eigensolution for Slightly Damped Systems

Assuming that the entries of the damping matrix c are small relative to those of the matrices m, g, and k, the matrix A can be written in the form

$$A = A_0 + A_1 \tag{A1}$$

where, from Eqs. (4b) and (8a)

$$A_0 = -L^{-1}K_0L^{-T} \qquad A_1 = -L^{-1}K_1L^{-T} \tag{A2}$$

in which

$$K_0 = \begin{bmatrix} g & k \\ -k & 0 \end{bmatrix} \qquad K_1 = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \tag{A3}$$

Observing that K_0 is skew symmetric and K_1 is symmetric, we conclude that A_0 is skew symmetric and A_1 is symmetric. The object is to produce a perturbation solution of the eigenvalue problem for A based on the eigensolution for A_0, on the assumption that the entries of A_1 are one order of magnitude smaller than those of A_0. Such a solution was obtained in Ref. 13, so that here we shall only summarize the results.

Assuming that A has the form given by Eq. (A1), we shall seek formal eigensolutions for the matrix A and A^T in the form

$$\lambda_j = \lambda_{0j} + \lambda_{1j} + \lambda_{2j} + \cdots$$
$$u_j = u_{0j} + u_{1j} + u_{2j} + \cdots$$
$$v_j = v_{0j} + v_{1j} + v_{2j} + \cdots \tag{A4}$$

where λ_{0j}, u_{0j}, v_{0j} will be referred to as the zero-order eigensolution, λ_{1j}, u_{1j}, v_{1j} as the first-order perturbation, λ_{2j}, u_{2j}, v_{2j} as the second-order perturbation, etc. In practice, it is seldom necessary to go beyond the second-order perturbation. The zero-order eigensolution, which is simply the eigensolution for A_0, can be obtained by some very efficient algorithms and is considered as known. Then, the various perturbations are computed on the basis of the zero-order eigensolution.

Because the matrix A_0 is real and skew symmetric, its eigenvalues occur in pairs of pure imaginary complex conjugates, $\lambda_{0j} = i\omega_j, \lambda_{0,j+n} = \bar{\lambda}_{0j} = -i\omega_j$ $(j = 1,2,\ldots,n)$, where ω_j can be identified as the undamped system natural frequencies. The associated right and left eigenvectors are u_{0j}, $u_{0,j+n} = \bar{u}_{0j}$ and $v_{0j} = \bar{u}_{0j}$, $v_{0,j+n} = \bar{v}_{0j} = u_{0j}$, respectively. It follows that it is not necessary to solve the eigenvalue problem for A_0^T. As shown in Ref. 13, if k is positive definite, it is possible to produce the eigensolution for A_0 in terms of real quantities alone by solving the eigenvalue problem for $-A_0^2$, where $-A_0^2$ is a real symmetric positive definite matrix. Computational algorithms for the eigensolution for real symmetric positive definite matrices are by far the most efficient and accurate of all eigensolution algorithms.

Based on the zero-order solution, the first-order perturbation has the form [13]

$$\lambda_{1j} = \tfrac{1}{2} u_{0j}^H A_1 u_{0j} \qquad u_{1j} = \sum_{k=1}^{2n} \epsilon_{ik} u_{0k} \qquad v_{1j} = -\bar{u}_{1j} \qquad (j=1,2,\ldots,2n) \tag{A5}$$

in which

$$\epsilon_{ik} = \frac{u_{0k}^H A_1 u_{0i}}{2(\lambda_{0i} - \lambda_{0k})} \qquad (i,k=1,2,\ldots,2n;\ i \neq k)$$

$$\epsilon_{kk} = 0 \qquad (k=1,2,\ldots,2n) \tag{A6}$$

where $u_{0j}^H = \bar{u}_{0j}^T$. We note that λ_{1j} are real and negative which is to be expected. Indeed, to the first approximation, small damping should introduce a negative real part into the eigenvalues. Moreover, the second-order perturbation has the form [13]

$$\lambda_{2j} = \sum_{k=1}^{2n} (\lambda_{0k} - \lambda_{0j}) \epsilon_{jk} \epsilon_{kj} \qquad u_{2j} = \sum_{k=1}^{2n} \tilde{\epsilon}_{jk} u_{0k} \qquad v_{2j} = \bar{u}_{2j} \qquad (j=1,2,\ldots,2n) \tag{A7}$$

in which

$$\tilde{\epsilon}_{jk} = \frac{1}{\lambda_{0j} - \lambda_{0k}} \left[(\lambda_{1k} - \lambda_{1j}) \epsilon_{jk} + \sum_{l=1}^{2n} (\lambda_{0l} - \lambda_{0k}) \epsilon_{jl} \epsilon_{lk} \right] \qquad (j,k=1,2,\ldots,2n;\ j \neq k)$$

$$\tilde{\epsilon}_{jj} = \frac{1}{2} \sum_{k=1}^{2n} \epsilon_{jk} \epsilon_{kj} \qquad (j=1,2,\ldots,2n) \tag{A8}$$

We observe that λ_{2j} are pure imaginary which is also to be expected, as to the second approximation, small damping should modify the undamped natural frequencies. Although Eqs. (A5-A8) may involve complex quantities, the actual computations involve real quantities alone.

Returning to Eq. (14b), we can identify its entries as follows:

$$\alpha_r = \lambda_{1r} \qquad \beta_r = \omega_r - i\lambda_{2r} \qquad (r=1,2,\ldots,n) \tag{A9}$$

Moreover, computing the eigenvectors $u_j \cong u_{0j} + u_{1j} + u_{2j}$, $u_{j+n} = \bar{u}_j$ and $v_j \cong v_{0j} + v_{1j} + v_{2j}, v_{j+n} = \bar{v}_j$ $(j=1,2,\ldots,n)$ and separating the real and imaginary parts of the vectors, it is possible to construct the matrices U and V, Eqs. (12), as well as the matrices P and Q, Eqs. (15).

For sufficiently small damping, the eigenvalues $\lambda_j = \alpha_j + i\beta_j$, $\lambda_{j+n} = \alpha_j - i\beta_j$ $(j=1,2,\ldots,n)$ and the matrices U and V produced by the foregoing scheme are very accurate. Moreover, because the computations only require an eigensolution of a positive definite real symmetric matrix, the foregoing scheme can produce eigensolutions for damped systems of orders exceeding by far the order that can be handled by algorithms for arbitray matrices.

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